RMSC 4003
Statistical Modeling in Financial Markets
Tutorial 2 Solution

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1 Asset Return

Using notations in lecture notes and assuming all vectors are column vectors:

(1) Simple return

$$r_t = \frac{P_t - P_{t-1} + D_t}{P_{t-1}}.$$

(2) Log-return

$$r_t^{\log} = \log \frac{P_t}{P_{t-1}} = \log P_t - \log P_{t-1}.$$

(3) Portfolio return

$$r_p = \sum_{i=1}^{n} w_i r_i = w' r \text{ where} \sum_{i=1}^{n} w_i = w' 1_n = 1.$$

Remark 1.1. log will always mean ln unless otherwise specified.

2 Portfolio Mean and Variance

Note that returns are random variables. Denote $\mathrm{E}(\mathbf{r_P}) = \mu = (\mu_\mathbf{1}, \ldots, \mu_\mathbf{n})'$ and $\mathrm{Var}(\mathbf{r_P}) = \mathbf{\Sigma} = (\sigma_\mathbf{ij})_{\mathbf{n}\times\mathbf{n}}$ where $\sigma_{ij} = \mathrm{Cov}(r_i, r_j)$.

(1) Mean return of a portfolio

$$\mu_P = \mathrm{E}(r_P) = \sum_{i=1}^{n} w_i \,\mathrm{E}(r_i) = w' \mu.$$

(2) Variance of portfolio return

$$\sigma_P^2 = \mathrm{E}[(r_P - \mu_P)^2] = \sum_{i,j=1}^{n} w_i w_j \sigma_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} = w' \Sigma w.$$

Recall that $\mathrm{Var}(aX + bY) = a^2 \mathrm{Var}\, X + b^2 \mathrm{Var}\, Y + 2ab \,\mathrm{Cov}(X, Y)$. Therefore, the formula for the variance of portfolio return can be easily deduced. In fact, using matrix notation, $\mathrm{Var}(w'X) = w' \mathrm{Var}(X) w$.

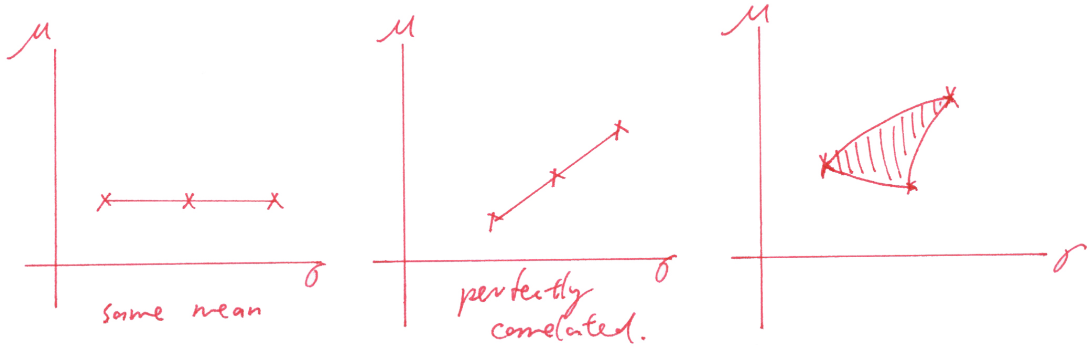
Remark 2.1. Without further mentioning the definition, r_p, r_A and r_B will usually be used to denote the returns of the portfolio P, asset A and asset B. Their corresponding mean and standard deviation will be denoted by $\mu_P, \mu_A, \mu_B, \sigma_P, \sigma_A, \sigma_B$.

3 Feasible Set

The set of points corresponding to possible portfolios that investors can choose is called the **feasible set** or **feasible region**. It satisfies the following two important properties:

(1) If there are at least three assets (not perfectly correlated and with different means), the feasible set will be a solid two-dimensional region.

(2) The feasible region is convex to the left, meaning that given any two points in the region, the straight line connecting them does not cross the left boundary of the feasible set.



4 Efficient Frontier

Efficient Set Theorem: An investor will choose a portfolio from the set of portfolios that

(1) offers maximum expected return for varying level of risk (**Non-satiation property**), and

(2) offers minimum risk for varying levels of expected return (**Risk aversion property**).

- **Minimum-variance Set**: Set of portfolios meeting the second condition
- **Efficient frontier** (or **Efficient Set**): The set of portfolios meeting the above two conditions
- **Minimum-variance Point**: The portfolio with the smallest variance in the feasible set

5 Indifference Curves

Indifference Curves: A curve satisfying an investor's preference to risk in the $\mu - \sigma$ diagram. Features:

- All portfolios lying on a given indifference curve are equally desirable to an investor.
- Indifference curve that lies further northwest is more desirable than those that are not as northwest.
- A more risk-averse investor has more steeply sloped ICs.

6 Two-Stock Example

Suppose there are only two stocks A and B with $\mu_A \neq \mu_B$ and correlation ρ. Assume there is no short selling so that $\alpha \in [0,1]$. Why the feasible set is a triangle? Since

$$r_P = \alpha r_A + (1-\alpha) r_B,$$

we have

$$\begin{aligned} \mu_P &= \alpha\mu_A + (1-\alpha)\mu_B \\ \sigma_P^2 &= \alpha^2\sigma_A^2 + (1-\alpha)^2\sigma_B^2 + 2\alpha(1-\alpha)\sigma_A\sigma_B\rho. \end{aligned}$$

When $\rho = 1$,

$$\sigma_P^2 = (\alpha\sigma_A + (1-\alpha)\sigma_B)^2.$$

This implies that $\sigma_P = \alpha\sigma_A + (1-\alpha)\sigma_B$ and so the portfolio will lie on the straight line connecting (σ_A, μ_A) and (σ_B, μ_B). Indeed, solving for α, we have

$$\alpha = \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}.$$

Then

$$\begin{aligned} \mu_P &= \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}\mu_A + \frac{\sigma_A - \sigma_P}{\sigma_A - \sigma_B}\mu_B \\ &= \left(\frac{\mu_A - \mu_B}{\sigma_A - \sigma_B}\right)\sigma_P + \frac{\mu_B\sigma_A - \sigma_B\mu_A}{\sigma_A - \sigma_B}, \end{aligned}$$

which is the equation of straight line passing through (σ_A, μ_A) and (σ_B, μ_B).

When $\rho = -1$, $\sigma_P^2 = (\alpha\sigma_A - (1-\alpha)\sigma_B)^2$. So, we have

$$\sigma_P = \begin{cases} \alpha\sigma_A - (1-\alpha)\sigma_B & \text{if } \alpha \geq \frac{\sigma_B}{\sigma_A+\sigma_B} \\ (1-\alpha)\sigma_B - \alpha\sigma_A & \text{if } \alpha < \frac{\sigma_B}{\sigma_A+\sigma_B} \end{cases}$$

For $\sigma_P = \alpha\sigma_A - (1-\alpha)\sigma_B$, $\alpha = \frac{\sigma_P+\sigma_B}{\sigma_A+\sigma_B}$. Therefore,

$$\begin{aligned} \mu_P &= \frac{\sigma_P + \sigma_B}{\sigma_A + \sigma_B}\mu_A + \frac{\sigma_A - \sigma_P}{\sigma_A + \sigma_B}\mu_B \\ &= \left(\frac{\mu_A - \mu_B}{\sigma_A + \sigma_B}\right)\sigma_P + \left(\frac{\sigma_A\mu_B + \sigma_B\mu_A}{\sigma_A + \sigma_B}\right), \end{aligned}$$

which is the equation of straight line passing through (σ_A, μ_A) and $(0, \frac{\sigma_A\mu_B+\sigma_B\mu_A}{\sigma_A+\sigma_B})$.

For $\sigma_P = (1-\alpha)\sigma_B - \alpha\sigma_A$, $\alpha = \frac{\sigma_B-\sigma_P}{\sigma_A+\sigma_B}$. Therefore,

$$\begin{aligned} \mu_P &= \frac{\sigma_B - \sigma_P}{\sigma_A + \sigma_B}\mu_A + \frac{\sigma_A + \sigma_P}{\sigma_A + \sigma_B}\mu_B \\ &= \left(\frac{\mu_B - \mu_A}{\sigma_A + \sigma_B}\right)\sigma_P + \left(\frac{\sigma_A\mu_B + \sigma_B\mu_A}{\sigma_A + \sigma_B}\right), \end{aligned}$$

which is the equation of straight line passing through (σ_B, μ_B) and $(0, \frac{\sigma_A\mu_B+\sigma_B\mu_A}{\sigma_A+\sigma_B})$.

Finally, given $\mu_P \in [\mu_A, \mu_B]$, α is fixed. $\sigma_P(\rho)$ is a continuous and increasing function on $[-1,1]$. Thus, the shape of the feasible set is a triangle.

Question: what is the shape of the feasible set if $\mu_A = \mu_B$? It will be a straight line connecting the two assets.

Remark 6.1. When ρ and α are unknown, the feasible set is a triangle. When ρ is given, the feasible set is a curve or a(two) line(s) when $\rho = 1 (\rho = -1)$.

7 Markowitz Model

Suppose there are n assets. The mean returns are $\mu_1, \dots, \mu_n$ and the covariances are σ_{ij} for $i, j = 1, \dots, n$. A portfolio is defined by a set of n weights $w_i, i = 1, \dots, n$ that sum to 1. (We allow negative weights, corresponding to short selling, in this setting.) To find the minimum variance portfolio with a given mean value μ_p. The problem becomes

$$\text{minimize } \frac{1}{2}\sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

$$\text{subject to } \begin{cases} \sum_{i=1}^{n} w_i \mu_i = \mu_p \\ \sum_{i=1}^{n} w_i = 1. \end{cases}$$

(The introduction of the term 1/2 is to simplify the calculation only.) We can use Lagrange multipliers methods to obtain the solution. Let λ_1 and λ_2 be the Lagrange multipliers and form the Lagrangian

$$L = \frac{1}{2}\sum_{i,j=1}^{n} w_i w_j \sigma_{ij} - \lambda_1(\sum_{i=1}^{n} w_i \mu_i - \mu_p) - \lambda_2(\sum_{i=1}^{n} w_i - 1).$$

By taking partial differentiation w.r.t. each variable w_i and setting these derivatives to zero, we will obtain n linear equations. Together with the two constraints, we can solve $n + 2$ unknowns by the following $n + 2$ equations:

$$\begin{aligned} \sum_{j=1}^{n} w_j \sigma_{ij} - \lambda_1 \mu_i - \lambda_2 &= 0, \quad i = 1, \dots, n \\ \sum_{i=1}^{n} w_i \mu_i &= \mu_p \\ \sum_{i=1}^{n} w_i &= 1. \end{aligned}$$

That is, to solve $Ax = b$ where

$$x = \begin{pmatrix} w_1 \\ \vdots \\ w_n \\ -\lambda_1 \\ -\lambda_2 \end{pmatrix} \quad A = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} & \mu_1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn} & \mu_n & 1 \\ \mu_1 & \mu_2 & \dots & \mu_n & 0 & 0 \\ 1 & 1 & \dots & 1 & 0 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mu_p \\ 1 \end{pmatrix}.$$

8 Examples

Example 8.1. Which of the following best describe the shape of the portfolio possibilities curve?

(a) The curve is strictly convex.

(b) The curve is strictly concave.

(c) The curve is concave above the minimum variance portfolio and convex below the minimum variance portfolio.

(d) The curve is convex above the minimum variance portfolio and concave below the minimum variance portfolio.

Ans: C.

Example 8.2. Show that

$$\frac{\partial}{\partial w_k}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}\right) = 2\sum_{j=1}^{n} w_j \sigma_{kj}.$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial w_k}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}\right) &= \frac{\partial}{\partial w_k}\left(w_1\sum_{j=1}^{n} w_j\sigma_{1j} + \ldots + w_n\sum_{j=1}^{n} w_j\sigma_{nj}\right) \\
&= \sum_{j=1}^{n} w_j\sigma_{kj} + w_1\sigma_{ik} + \ldots + w_n\sigma_{nk} \\
&= \sum_{j=1}^{n} w_j\sigma_{kj} + \sum_{i=1}^{n} w_i\sigma_{ik} \\
&= 2\sum_{j=1}^{n} w_j\sigma_{kj}. \quad (\text{since } \sigma_{ij} = \sigma_{ji})
\end{aligned}$$

□

Example 8.3. Suppose that there are three assets with the following mean and variance-covariance matrix:

$$\mu = \begin{pmatrix} 0.07 \\ 0.12 \\ 0.09 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.25 \end{pmatrix}$$

Consider the portfolio with mean return $\mu_p = \sum_{i=1}^{3} w_i\mu_i$.

(a) Using these values, write down the system of equations for the Markowitz model to determine w_i's.

(b) Solve w_i's in terms of μ_p from the above system of equations.

(c) What is the variance of this portfolio?

(d) What is the global minimum variance portfolio?

Solution. (a)

$$\begin{cases} 0.2w_1 - 0.07\lambda_1 - \lambda_2 = 0 & (1) \\ 0.3w_2 - 0.12\lambda_1 - \lambda_2 = 0 & (2) \\ 0.25w_3 - 0.09\lambda_1 - \lambda_2 = 0 & (3) \\ 0.07w_1 + 0.12w_2 + 0.09w_3 = \mu_P & (4) \\ w_1 + w_2 + w_3 = 1 & (5). \end{cases}$$

(b) (1)-(2) implies

$$0.2w_1 - 0.3w_2 + 0.05\lambda_1 = 0.$$

(2)-(3) implies

$$0.3w_2 - 0.25w_3 - 0.03\lambda_1 = 0.$$

These two equations imply

$$\begin{aligned} 0.6w_1 + 0.6w_2 - 1.25w_3 &= 0 \\ w_1 + w_2 - \frac{25}{12}w_3 &= 0. \end{aligned}$$

Together with $w_1 + w_2 + w_3 = 1$, we know $w_3 = 12/37$ and $w_1 = 25/37 - w_2$. Therefore, from (4), we have

$$\begin{aligned} 0.07(25/37 - w_2) + 0.12w_2 + 0.09(12/37) &= \mu_P \\ w_2 &= 20\mu_P - 283/185. \end{aligned}$$

Hence,

$$\begin{cases} w_1 = -20\mu_P + \frac{408}{185} \\ w_2 = 20\mu_P - \frac{283}{185} \\ w_3 = 12/37. \end{cases}$$

(c)

$$\sigma_P^2 = w_1^2(0.2) + w_2^2(0.3) + w_3^2(0.25) = 200\mu_P^2 - 36\mu_P + \frac{3147}{1850}.$$

(d)

$$\frac{d\sigma_P^2}{d\mu_P} = 400\mu_P - 36 \Longrightarrow \mu_P = 0.09.$$

Hence, the global minimum variance portfolio is given by the weights $(\frac{15}{37}, \frac{10}{37}, \frac{12}{37})$.